# Using the Fisher-Cramer-Rao bounds for evaluating the performance limits of EW techniques

J.C. Olivier

{corne.olivier}@up.ac.za
http://www.ee.up.ac.za/main/en/personnel/jolivier/index

Defense, Peace, Safety and Security, CSIR, Pretoria
Department of Electronic Engineering, University of Pretoria, Pretoria

August 2009

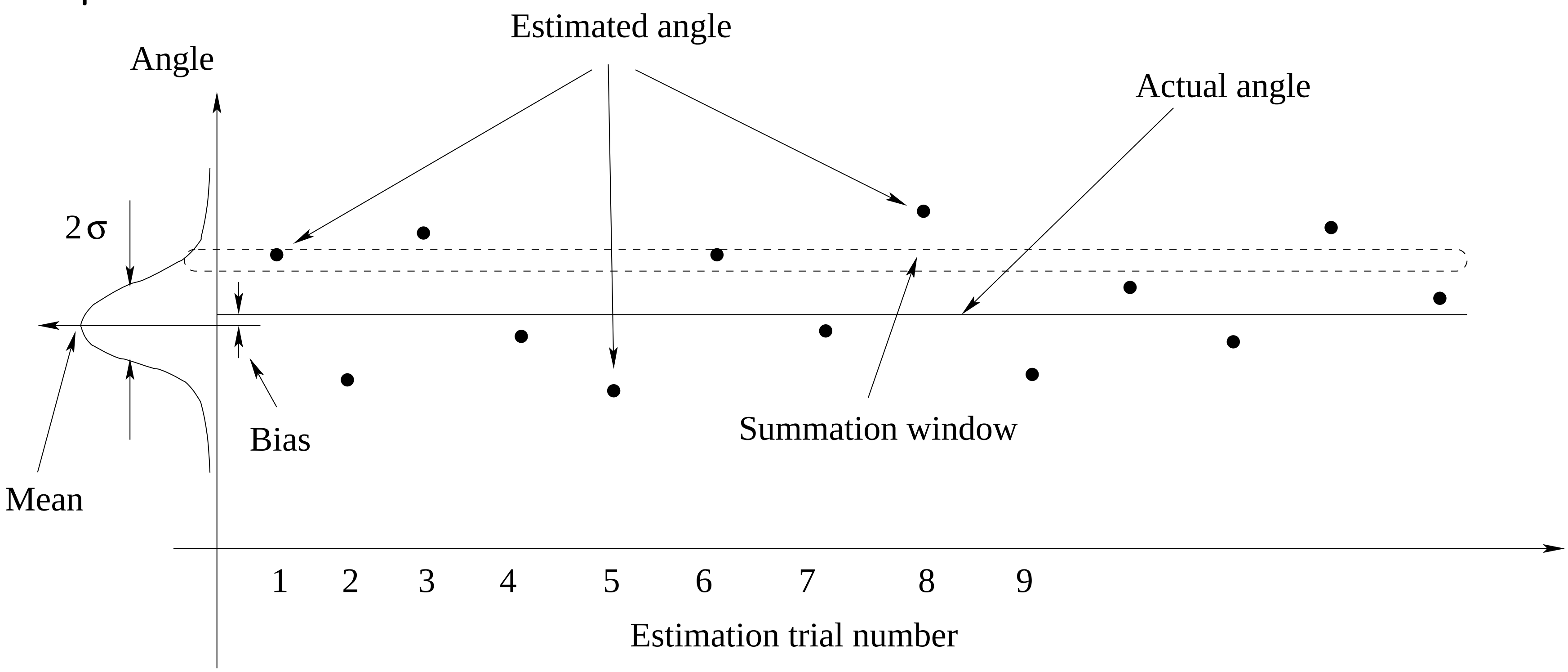
# Outline of presentation

- Introduction to the concept of estimator optimality
- The estimator variance and Cramer-Rao Bound (CRB)
- Example: Estimating the Angle of Arrival of a pulse using an antenna array
- The MUSIC method for Angle of Arrival and the CRB
- Conclusions

# Ronald Fisher

# Introduction to the concept of estimator optimality

- Consider an estimator for estimating the angle of arrival of a plane wave



# Introduction to the concept of estimator optimality

- We are presented noisy measurement data denoted $x[0], x[1], \cdots, x[N-1]$
- We wish to estimate parameters $\theta$ given $\mathbf{x}$
- There may be one or more sensors available, say $M$
- As $M \rightarrow \infty$ and/or $N \rightarrow \infty$, $Pr(|\widehat{\theta} - \theta| > \epsilon) = 0$ i.e. consistency
- Quality measured by *mean square error* (MSE) $= E\{(\widehat{\theta} - \theta)^2\}$
- This can be expanded as $MSE = E\{[\widehat{\theta} - E(\theta)]^2\} + (E(\theta) - \theta)^2$
- Term 1: Variance and term 2: Bias squared

# The minimum possible variance: the Cramer-Rao Bound (CRB)

Theorem: (Cramer-Rao-Fisher) *Given the SNR, the number of observations and the hardware constraints, the variance (or standard deviation) of the estimate of* $\theta$ *yielded by* any *unbiased estimator is at least as high as the inverse of the Fisher information* $I(\theta)$.

Fisher Information matrix $I(\theta)$: Let $\theta$ distributed according to probability density function $f(x;\theta)$, $x$ denote the measurements.

$$I(\theta) = E\left[\left(\frac{\partial log f(x,\theta)}{\partial\theta}\right)^2\right]$$

Minimum possible variance of ANY estimator is inverse of $I(\theta)$

$$var(\widehat{\theta}) \geqslant \frac{1}{I(\theta)}$$

# Cramer-Rao bounds for a class of non-linear estimators: MUSIC

According to the MUSIC paradigm estimation problems can be cast as

$$y = A(\theta)x(t) + e(t) \quad \forall \;\; t = 1, \cdots, N$$

$y$ is a column vector of m elements, $x(t)$ is a column vector with n elements, $A(\theta)$ is a $(m, n)$ matrix and $e$ is a column vector of additive noise components with known distribution.

- Estimating the carrier frequency of a pulse can be written in this form
- Estimating the angle of arrival (AOA) of plane waves on an array can be written in this form

# Cramer-Rao bounds for a class of non-linear estimators: MUSIC

The CRB for this class of estimation problems can be shown to be given by (see Petre Stoica 1989)

$$CRB(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{t=1}^{N} Re\{X^\dagger(t) D^\dagger [I - A(A^\dagger A)^{-1} A^\dagger)] D X(t)\} \right\}^2$$

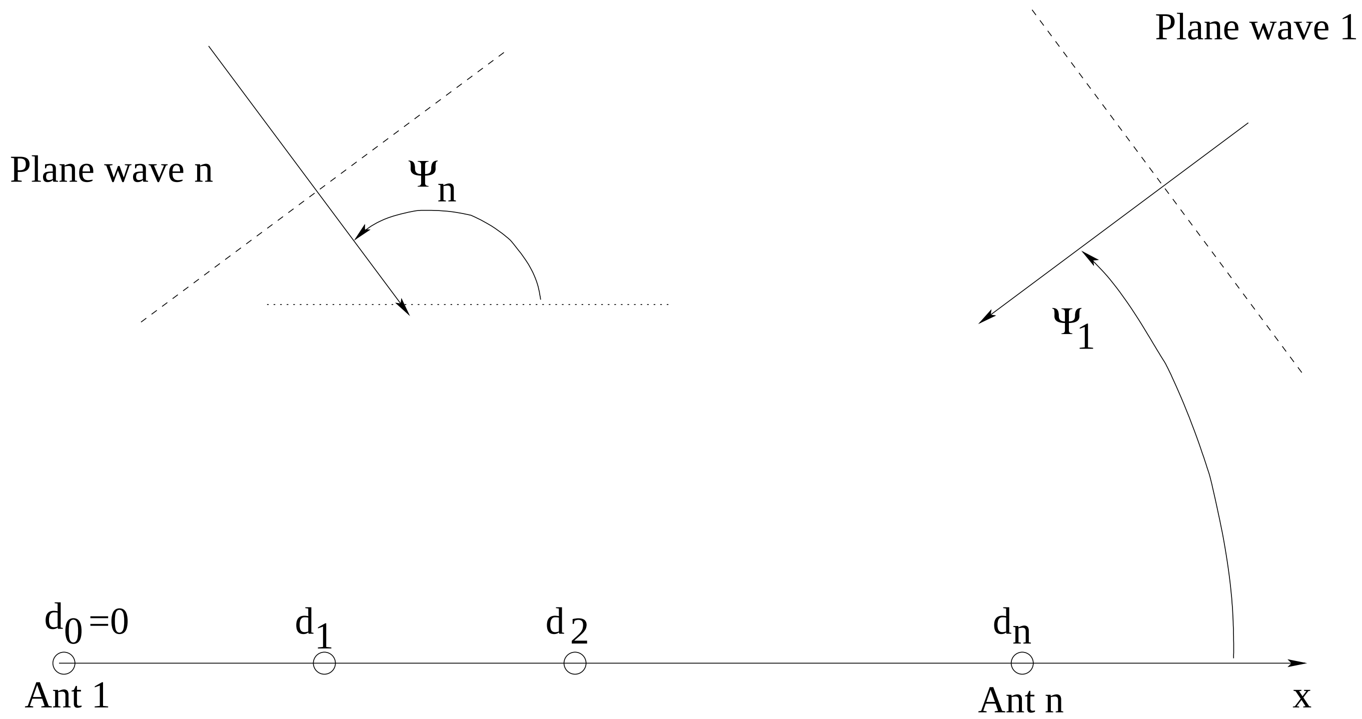
$$X(t) = \begin{bmatrix} x_1(t) & 0 & \cdots & 0 \\ 0 & x_2(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & x_n(t) \end{bmatrix}$$

$$D = [\frac{\partial a_1(\omega_1)}{\partial \omega_1}, \cdots, \frac{\partial a_n(\omega_N)}{\partial \omega_N}]$$

where $a_i$ is the i'th column of $A$ and $\theta$ has elements $\omega$.

# Angle of arrival estimation using a linear array

The problem considered is to estimate the Angle of Arrival (AOA) for multiple plane waves with different frequencies using a linear array.



# Angle of arrival estimation using a linear array: MUSIC

Consider a 3 antenna array uniformly spaced at $d$ meters, 2 plane waves incident. A mathematical model for the MUSIC formulation is as follows:

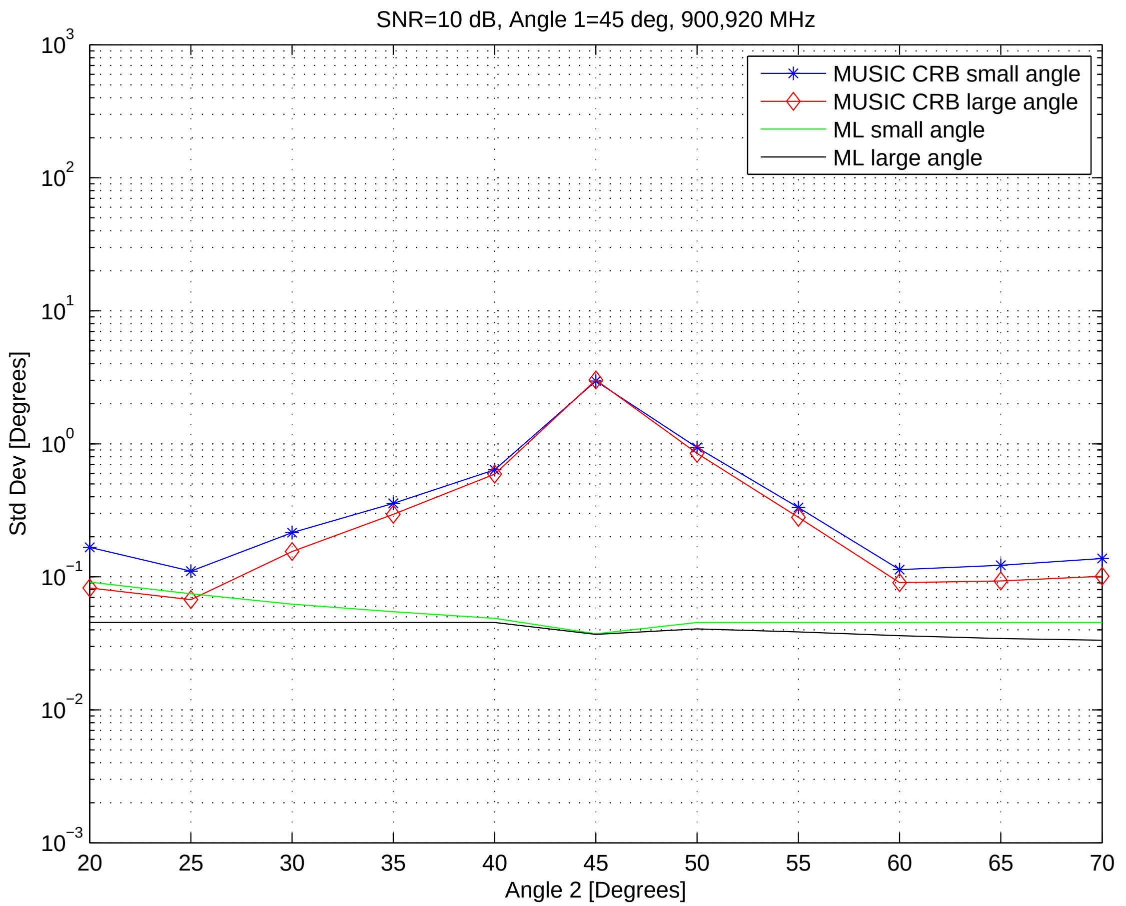
$$\begin{bmatrix} y_1^t \\ y_2^t \\ y_3^t \end{bmatrix} = \begin{bmatrix} e^{j\frac{2\pi f d 0 cos(\psi_1)}{c}} & e^{j\frac{2\pi f d 0 cos(\psi_2)}{c}} \\ e^{j\frac{2\pi f d 1 cos(\psi_1)}{c}} & e^{j\frac{2\pi f d 1 cos(\psi_2)}{c}} \\ e^{j\frac{2\pi f d 2 cos(\psi_1)}{c}} & e^{j\frac{2\pi f d 2 cos(\psi_2)}{c}} \end{bmatrix} \begin{bmatrix} A_1 e^{j2\pi f_1 t} \\ A_2 e^{j2\pi f_2 t} \end{bmatrix} + \begin{bmatrix} e_1^t \\ e_2^t \\ e_3^t \end{bmatrix}$$

# Angle of arrival estimation using a linear array: MUSIC

Consider a 3 antenna array.

- 2 GHz sample rate and both sources are pulses
- Antennas spaced at $d = [0, 0.15, 1.5]$ meters
- 1 GHz bandwidth in baseband
- 2 plane waves (sources) incident, 900 MHz and 920 MHz
- One source is kept at 45 degrees fixed, other one is varied.
- 200 samples available
- SNR = 10 dB

# Angle of arrival estimation using a linear array: MUSIC



# Conclusions

- Cramer-Rao bound theory provides the lower bounds for estimator performance that can be achieved in practice given hardware constraints

- All estimators that can be devised must be compared to the Cramer-Rao bound limits to determine their *optimality* - i.e. how good they really are

- Bias can be traded for variance, but both can only be reduced by increasing hardware cost or number of measurements